THE IMPROVEMENT OF ABILITY IN THE USE OF THE FORMAL OPERATIONS OF ALGEBRA BY MEANS OF FORMAL PRACTICE EXERCISES

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SUGGESTIONS AND DIRECTIONS FOR THE USE OF SUCH MATERIAL IN A CO-OPERATIVE EXPERIMENT IN THE TEACHING OF FIRST-YEAR ALGEBRA, CARRIED ON WITH A NUMBER OF PROGRESSIVE TEACHERS SEEKING TO IMPROVE THE TEACHING OF THE SUBJECT

Two important problems face every teacher of algebra as that subject is organized and taught at the present time: (1) the determination of effective ways of rationalizing the processes of algebra and of developing facility in the use of the most important algebraic devices—the equation, the formula, and the graph; (2) the determination of *economical* methods of teaching the formal operations to the extent that pupils in first-year algebra will have *satisfactory* skill in manipulating them. The general problem of improving the teaching of first-year algebra has been contributed to by the writers' research of the past four years. This classroom research has been based on the assumption that to aid in solving the general problem most completely we must set up an experimental program involving the following five steps:

1. Stating definitely the aims and outcomes of instruction in first-year algebra.

2. Classifying clearly the subject-matter of the course on a basis of the principal modes of learning involved in its mastery; a statement of the content of the course, from a standpoint both of the amount of material included and of the classification, arrangement, and order of presentation, supplemented by a detailed analysis of the mental processes called into play by different types of subject-matter.

3. *Designing and giving tests which will adequately measure ability in each of the fundamental phases of the subject-matter agreed upon.*

4. Critically evaluating the testing so as to give a complete and differentiated statement of fundamental weaknesses in learning (e.g., as revealed by the typical errors made by pupils).

5. Setting up experimental attempts to eliminate these fundamental weaknesses; from the standpoint of economy of time this means the determination of "best methods" of *rationalizing* these operations, *the design of practice exercises, the determination of specific "best ways" of presenting material, of order of presentation, and of optimum length of drills.*[1]

In an intensive investigation during the past four years the writers have carried through to tentative conclusion the first four steps in the foregoing program and are now initiating work on the fifth and most important step. A word of caution should be stated at this point: *We insist that the fundamental aim of instruction in first-year algebra is to develop in the pupil the ability to use intelligently the most powerful devices of quantitative thinking:* the EQUATION, the FORMULA, and the GRAPH. Only in so far as habituation or automatic efficiency in the formal processes contributes to efficiency in the solution of problems primarily of the reasoning or interpretive type do we insist upon skill or habituation in them. But there is evidence, both of expert opinion and of experimentation, to show that ability in the solution of new and original problems is accompanied by, and varies roughly with, *the degree to which the pupil has habituated these essential tool operations* (see article in *School Review*, February and March, 1917). We recognize, therefore, that this phase of the teaching process is only subsidiary to, although a *necessary* correlate of, the more fundamental aim stated above.

Carefully controlled tests of ability of pupils in first-year algebra, covering more than fifty school systems, have revealed: first, a very decided lack of skill in the manipulation of the formal operations; secondly, evident inability to use these operations successfully in their applications. It should be stressed that we recognize that the most important outcomes of the study of algebra are

[1] Both of the present writers are now carrying on, under typical public-school conditions, a classroom experiment of this sort. The results will be reported to readers of this journal several times during the next two years.

thought outcomes. We insist, therefore, that in order to give proper emphasis to this thought aspect *we must devise ways and means of reducing the undue amount of time now being given to the tool processes, in order to spend more time upon their application.* That is, our fundamental problem is both a problem of teaching emphasis and of proper distribution of time on the various aspects of teaching. Thus our immediate aim is to organize a teaching program that will result in increasing the pupil's efficiency in the use of the necessary tool operations with the expenditure of such a small amount of time that by far the larger proportion of time and energy can be put on the interpretive aspect of teaching.

The results of the testing and of the preliminary classroom experimentation of the past year have shown clearly that mass-instruction of the practice or drill types can be effectively given by means of formal practice exercises. Also that it will result in a very satisfactory proficiency in the tool operations with the expenditure of a relatively small proportion of the total class time. It should be stressed that formal drill can be given by means of this material without the use of more than ten minutes to fifteen minutes per day. Having experimented carefully with the use of tests and practice exercises the writers now wish to put these before teachers of algebra in the hope that they may aid in solving one of the important problems of mathematics teaching.

The purposes of the practice exercises[1] should be made clear at the start. They may be listed definitely as follows:

PURPOSES OF PRACTICE EXERCISES

1. In general, to make the pupil more efficient in his skill in manipulating the "tool" operations of first-year algebra.

2. To concentrate practice upon those operations that have been shown to be particularly difficult for the pupil, i.e., those types of problems in which the pupil makes the greatest number of mistakes (see table of typical difficulties in article in March *School Review*, referred to above).

3. To set up for the pupil and the teacher a definite, objective goal of achievement to be attained, *after which further practice will probably not yield results commensurate with the time required.*

[1] One card of the seven in the complete set is reproduced on p. 549.

4. To employ as a teaching device one of the most influential factors of learning, namely, short, spirited, frequent, "*timed*," practice periods.

STANDARDIZED PRACTICE EXERCISES
FIRST-YEAR ALGEBRA

Set. No. 9. Factoring continued
Standard: 12 "rights" in 6 min.

1. $3x^2-6x-24$.....
2. $9x^4-25y^2$.....
3. p^4-p^2-20.....
4. $p^2+5p+10$.....
5. $9x^2+12xy+4y^2$.....
6. $2y^2-6y-20$.....
7. $16r^6-49y^4$.....
8. x^6-x^3-12.....
9. c^2+3c+8.....
10. $25d^2+30dc+9c^2$.....
11. $5a^2-20a-60$.....
12. z^4-81w^6.....
13. y^8-y^4-30.....
14. $d^2+7d+11$.....
15. $49p^2+28pq+4q^2$....
16. $3b^2-15b-150$.....
17. $4n^{10}-121x^8$.....
18. x^2-x-56.....

Set. No. 10. Quadratic Equations
Standard: 12 "rights" in 3 min.

1. $x^2-2x=8$.....
2. $b^2-b=6$.....
3. $x^2-36=0$.....
4. $n^2-5n-6=0$.....
5. $s^2+4s=21$.....
6. $b^2-4b=45$.....
7. $y^2-y=20$.....
8. $r^2-49=0$.....
9. $t^2-7t-8=0$.....
10. $x^2+6x=40$.....
11. $n^2-8n=33$.....
12. $p^2-p=12$.....
13. $s^2-25=0$.....
14. $y^2-9y-10=0$.....
15. $a^2+3a=54$.....
16. $x^2-7x=44$.....
17. $c^2-c=30$.....
18. $m^2-64=0$.....

5. To stimulate the pupil to compete against his own previous record, made definite and objective, and kept by himself as a measure of his own progress in the mastery of the subject-matter.

6. To help reduce, as much as possible, the great amount of time now being given to class drills, which careful measurement and individual experience show to be ineffective.

7. To provide a means by which individual differences may be practically recognized and met.

GENERAL PRINCIPLES OF DESIGN OF PRACTICE EXERCISES

1. We wish to make it clear that these practice exercises have been designed after detailed study of the *errors* of pupils. Pupils' errors are of two distinct types: (*a*) accidental errors, and (*b*) recurring errors. *It is the recurring errors that are significant* to the teacher who is trying to perfect the pupil's manipulation of the formal operations, and it is from a detailed study of these that the practice exercises have been designed. This study has enabled us to design exercises that would offer practice on the various operations somewhat closely in accordance with their relative difficulty. (The February and March [1917] numbers of the *School Review* contain a complete description of the work that has been done in testing pupils' abilities and in determining important types of errors.)

2. These exercises have been designed on the basis of the facts revealed by examination of the *achievements* of large numbers of pupils on the writers' "Standardized Algebra Tests."[1] That is, they have been designed to give the pupil practice on particular algebraic processes roughly in proportion to their difficulty, as shown by the number of pupils unable to solve them.

3. The limitations of space make a detailed explanation of the determination of the *standards* impracticable here. Suffice it to say—these standards are tentative and sufficiently low that approximately 50 per cent of the class will attain them on the fourth or fifth trial. The general principle underlying the establishment of the standards is that practice should not be extended beyond the point at which there ceases to be an increased efficiency, except for purposes of permanence or retention.

4. More problems have been included on each card than the number in the standard. They are arranged on a definite rotation scheme and *can be used as tests* if desired. This makes it feasible to begin with any problem in the list and work either way. By so doing the pupils are much less likely to remember the answer to particular problems.

5. Types of formal problems not necessary to the realization of the primary aim of instruction in first-year algebra stated above are *not* included here for practice. The writers question the advis-

[1] Published and distributed by the School of Education, University of Chicago, at 4 cents per set.

ability of such types of problems as division of polynomials by polynomials, factoring the sum and difference of two cubes, factoring by grouping terms, finding highest common factor, complicated combinations of fractions, etc. (list to be extended after further investigation).

SUGGESTIONS FOR THE USE OF THE PRACTICE EXERCISES

1. The foregoing discussion should make it clear that the practice exercises are in no wise intended to take the place of, nor to be given simultaneous with, the most thorough rationalization and explanation of the particular types of problems under consideration. They should be given *after* the class has clearly grasped the meaning and significance of the operation. The length of time devoted to getting this clear understanding of the various operations must necessarily vary with the teacher. Thus the time to begin the use of formal practice on any operation cannot be standardized for large groups of teachers. Each one must determine at what period he should turn to the use of the formal material.

2. We suggest that practice should be given on a particular exercise (e.g., the one on collecting terms if that operation has just been explained and discussed thoroughly) each day until 75 per cent of the class attains the standard number of problems set for that exercise, two days in succession. That is, excuse any individual pupil from practice on a particular exercise as soon as he has made the standard score twice in succession. Remember that these standard scores have come from the actual attainments of three large classes. They are set at the point attained by 50 per cent of the class on the third, fourth, or fifth trial, depending on the number of the exercise. If, after trying the scheme, the teacher is convinced that the standards are too low or too high, he may change them to suit his own purposes. The writers are convinced that skilful use of these exercises will tend to *raise* the standards in the future rather than to lower them.

3. The presence of rapid, average, and slow pupils in the same class necessitates a decision concerning the best use to make of the time of the rapid pupil. Acting on the principle that we want the "*optimum*" amount of *formal* skill, not the extreme amount of which the pupil is capable, we suggest excusing the pupil from the class

to work on problems of application or interpretation concerning those operations which have thus far been explained. That is, experience with both arithmetic and algebra exercises is proving that it is better to give the fast pupil *intensive* training with each operation, both "formal" and "applied," rather than to permit him to move on at once to the learning of other types of subject-matter. In this way *the members of the class are held together as regards formal skill* (which we emphasize as a fundamental principle) but *are permitted to get as much facility as they are capable of in the use of algebraic devices in thinking situations.* Thus the fundamental doctrine is: "limitation of training" in formal skills in order that the utmost possible opportunity may be given the bright pupil for developing his power to meet "problem" situations.

In order to excuse the rapid pupil from the practice exercise the teacher must have "problem" work of some kind at hand from which to make assignments. Teachers will do well to make a detailed study of this aspect of the general procedure, and will be amply repaid if they collect and classify carefully problems which will give varying opportunity for application and interpretation. The writers will be in a position shortly to make suggestions on this aspect of the teaching problem.

4. With the standard skill once attained on a given operation the *drill time* of the class goes most largely to a new operation. The teacher, however, faces the problem of giving "*review*" *practices* to "*hold the skill.*" The writers are now experimenting carefully in order to be able to make detailed suggestions as to the best intervals to leave between the practices. In the meantime the decision must be left largely to the initiative of the individual teacher. It is probable, however, that it will be helpful to review a given exercise under "time" conditions *as often as once a week.* On the whole it is important to make the use of the exercises very *flexible* and to fit their use to the needs and abilities of the individual class.

CLASSROOM PROCEDURE

In the specific classroom handling of the practice material the writers have found it helpful to keep in mind the following suggestions:

1. Economize time in passing and collecting cards. Let the first pupil in each row place the card for the day's practice on the desk of each pupil with the "*problem side*" *up*. Immediately at the close of the practice period let the last pupil in the row collect the cards in that row. Systematize the procedure from the first day.

2. Make all directions *before* the cards are passed. From the start impress it on the pupils that any study of the exercise on their desks before the teacher's signal "go" is unfair. The work of the pupil is to be done on a blank sheet of paper against which the practice card is placed. Caution the pupils *not* to write on the card itself.

3. To prevent memorizing answers to examples on successive days, change the number of the example with which the class begins; i.e., if they begin with the first example on the card today, tomorrow they may begin with the fourth, the sixth, or with the last one on the card and work backward toward the first.

4. With all pupils ready, start all to working at the signal (e.g., "go"). To be effective the exercise must be a "time-drill." It is of the utmost importance that the teacher time the exercise accurately. Always start the class at the even minute or half-minute by your watch. Long experience has shown us that it takes the greatest care to conduct these exercises effectively. Make sure that each pupil *stops exactly* at the signal "stop."

5. One of the most valuable outcomes of the use of the practice material is the part that the pupil takes in correcting the work, noting his mistakes and recording his own achievement and comparing it with the standard and with the achievements of other pupils in the same class. Thus systematize from the start the scoring, recording, and criticizing of the work done. Have the pupils exchange papers *immediately at the word "stop."* The answers to the examples are printed on the back of each card in the exact order of the examples themselves. The pupil turns his card over, lays the "answer side" against his neighbor's work and checks each answer. He then writes the number of "Attempts" and "Rights" on the paper and passes it back to the owner. The latter then records his score in "Attempts" and in "Rights" on

his *record card.* At irregular intervals and *always* when a pupil has made the standard score the teacher will need to check the accuracy of the pupil's work.

6. The teacher can stimulate interest in self-improvement by various devices. A helpful one is: have those who made the standard score hold up their hands, commend their work, and raise the question with the rest as to how many can do better the next day. The teacher will find that a *human* interest on his part in the pupils' scores will be a great stimulus to increased effort on their part.

7. Teachers should keep the practice cards and the record cards except during practice periods.

THE COST IN MONEY TO THE TEACHER WHO INVESTS IN THIS PRACTICE MATERIAL[1]

The writers believe that practice material in any experimental stage should *not* be commercialized. Their practice exercises have been designed after careful experimentation and great labor, in the firm belief that they will aid in improving the teaching of elementary algebra. The financial charge for these to the teacher has been set, therefore, at that minimum figure which will barely cover the cost to the writers for (1) printing, (2) handling, and (3) the expense of conducting the co-operative study next year. The charge worked out on this *cost basis* has been set at:

9 cents per set of 7 practice cards each
$\frac{1}{2}$ cent per pupil record card

A teacher in ordering will need: (1) *as many sets of practice cards as she has pupils in her largest class,* with perhaps a few more to cover wastage; (2) a pupil record card for *each different pupil* in her classes. Thus, if a teacher's largest class is 25 and she has 60 different pupils the entire cost to her will be $2.55. It is undoubtedly true that a set of cards will last two or three years—perhaps more. The foregoing charge can be reduced with future revised editions in case the sale of the exercises justifies a large printing. The present cost, small as it is, could have been reduced if it had been possible to venture a large printing of the set. It will be noted that these cards have been copyrighted. This has been done to protect them from premature commercialization and strictly in the interest of the scientific study of methods of improving teaching in algebra.

[1] The "Practice Exercises" are distributed by H. O. Rugg, School of Education, University of Chicago.